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Conjugate free convection on a vertical surface

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Abstract—It is shown that the equations governing the conjugate free convection boundary-layer flow on a vertical plate can be made dimensionless so as to involve only the Prandtl number. An efficient finite-difference scheme is developed to solve these equations and results are given for $Pr = 0.72$, 0.733 and 7.0, respectively. It is seen that an asymptotic expansion gives reliable results even at moderate values of x (dimensionless distance along the plate).

1. INTRODUCTION

The heat transfer characteristics of convective boundary-layer flows are strongly influenced by the form of the thermal boundary conditions imposed. It is usual to prescribe either the wall temperature or the wall heat flux, and a considerable amount of work has been done in understanding these flows over a wide range of wall conditions and fluid properties. Much of this work is reviewed in a recent book by Gebhart *et al.* [1]. There is another class of wall conditions in which there is an interaction between the convective fluid and conduction through the bounding wall. These are usually referred to as conjugate heat transfer problems, and much of the earlier work on this topic is also reviewed by Martynenko and Sokovishin [2].

More specifically, we assume that we have a vertical solid plate, at one side of which the temperature is maintained at a constant value T_0, while on the other side a natural convection boundary-layer flow is set up in fluid with constant ambient temperature T_∞ (assuming $T_0 > T_\infty$). The temperature on the convective surface is then determined by a balance between steady conduction through the plate and convective heat transfer from the plate.

This problem has been considered previously by Pozzi and Lupo [3], and by these authors for forced convection [4]. The method used by Pozzi and Lupo [3, 4] to solve the governing equations was by series expansion. This method has a serious drawback in that it is somewhat cumbersome to implement, without any guaranteed accuracy or even convergence and gives results which are unreliable even at moderate distances from the leading edge. Here, we modify the finite-difference scheme used successfully by the present authors for a wide range of convective boundary-layer flow calculations, see, for example, refs. [5–7]. This scheme was found to work easily and gave a simpler and more efficient method for obtaining a solution for a given value of the Prandtl number.

2. MATHEMATICAL FORMULATION

Consider the steady natural convection boundary-layer flow from a vertical plate of length l and thickness b. The outside surface of the plate is maintained at the constant temperature T_0, above the ambient temperature T_∞ of the convecting fluid. The physical situation is shown in Fig. 1.

On the assumption (consistent with boundary-layer theory) that axial conduction can be neglected, the equation for the temperature in the plate, T_s, is given by

$$\frac{\partial^2 T_s}{\partial \bar{y}^2} = 0, \quad 0 \leqslant \bar{x} \leqslant l, \quad -b \leqslant \bar{y} \leqslant 0. \tag{1}$$

Applying the condition that $T_s = T_0$ on $y = -b$ gives

$$T_s(\bar{x}, \bar{y}) = T_w(\bar{x}) + \frac{T_w(\bar{x}) - T_0}{b}\bar{y}, \tag{2}$$

where $T_w(\bar{x})$ is the temperature at the solid–fluid interface, and will be determined by the solution of the free convection problem. It should be noted that the same assumption that the axial heat conduction through the flat plate is insignificant has also been made by several authors, e.g. refs. [8–10]. The equations governing this convective flow are, on making the usual Boussinesq approximation,

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{3a}$$

NOMENCLATURE

b	thickness of the plate
g	acceleration due to gravity
k_s	thermal conductivity of the plate
k_f	thermal conductivity of the convecting fluid
l	length of the plate
Pr	Prandtl number
T	temperature in the convecting fluid
T_s	temperature in the plate
T_0	temperature (constant) at the outer edge of the plate
T_∞	ambient temperature (constant)
T_w	temperature at the interface between the plate and the convecting fluid
ΔT	applied temperature differences, $\Delta T = T_0 - T_\infty$
$\bar{u}$	velocity in the x direction (dimensionless as u)
$\bar{v}$	velocity in the y direction (dimensionless as v)
$\bar{x}$	co-ordinate along the plate (dimensionless as x)
$\bar{y}$	co-ordinate normal to the plate (dimensionless as y).

Greek symbols

β	coefficient of thermal expansion
θ	dimensionless temperature, $\theta = (T - T_\infty)/\Delta T$
θ_w	dimensionless plate temperature, $\theta_w = (T_w - T_\infty)/\Delta T$
ν	kinematic viscosity
ψ	(dimensionless) stream function.

Superscripts

$'$	differentiation with respect to ζ
$-$	dimensional variables.

Subscripts

w	wall condition
∞	ambient condition.

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = g\beta(T - T_\infty) + \nu\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \quad (3b)$$

$$\bar{u}\frac{\partial T}{\partial \bar{x}} + \bar{v}\frac{\partial T}{\partial \bar{y}} = \frac{\nu}{Pr}\frac{\partial^2 T}{\partial \bar{y}^2}, \quad (3c)$$

where $\bar{u}$ and $\bar{v}$ are the velocity components in the $\bar{x}$

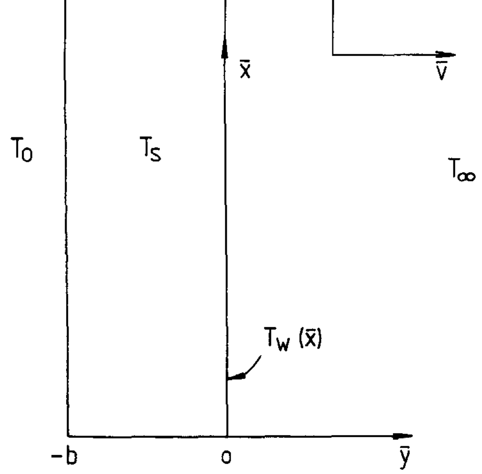

Fig. 1. The physical model and co-ordinate system.

and $\bar{y}$ directions, respectively, T is the temperature of the convecting fluid and ν and Pr are the kinematic viscosity and Prandtl number, respectively. The boundary conditions to be applied are that

$$\bar{u} \to 0, \quad T \to T_\infty \quad \text{as}\ \bar{y} \to \infty \quad (4a)$$

and, through continuity of temperature and heat flux at the solid–fluid boundary, that

$$T = T_w(\bar{x}), \quad k_s\frac{\partial T_s}{\partial \bar{y}} = k_f\frac{\partial T}{\partial \bar{y}} \quad \text{at}\ \bar{y} = 0 \quad (4b)$$

together with the no-slip condition that

$$\bar{u} = \bar{v} = 0 \quad \text{at}\ \bar{y} = 0. \quad (4c)$$

Here k_s and k_f are the thermal conductivities of the solid and the fluid, respectively. Using equation (2), condition (4b) gives

$$T = T_w, \quad \frac{\partial T}{\partial \bar{y}} = \frac{k_s}{bk_f}(T - T_0) \quad \text{on}\ \bar{y} = 0. \quad (5)$$

To make equation (3) and boundary condition (4a, c) and (5) dimensionless, we use the imposed temperature scale $\Delta T = T_0 - T_\infty$, as in ref. [3]. However, for a length scale we use a convective length scale $(g\beta\Delta T/\nu^2)(bk_f/k_s)^4$. This then leads us to write

$$T - T_\infty = \Delta T\theta, \quad \bar{u} = \frac{g\beta\Delta T}{\nu}\left(\frac{bk_f}{k_s}\right)^2 u, \quad \bar{v} = \frac{\nu k_s}{bk_f}v,$$

$$x = \frac{\nu^2}{g\beta\Delta T}\left(\frac{k_s}{bk_f}\right)^4 \bar{x}, \quad y = \frac{k_s}{bk_f}\bar{y}. \quad (6)$$

On using equation (6) and defining a dimensionless stream function ψ in the usual way, equations (3) become

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\,\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\theta+\frac{\partial^3\psi}{\partial y^3} \tag{7a}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}=\frac{1}{Pr}\frac{\partial^2\theta}{\partial y^2} \tag{7b}$$

with boundary conditions (4a, c) and (5) giving

$$\frac{\partial\psi}{\partial y}\to 0,\quad \theta\to 0\quad \text{as}\, y\to\infty \tag{7c}$$

$$\psi=0,\quad \frac{\partial\psi}{\partial y}=0,\quad \frac{\partial\theta}{\partial y}=\theta-1\quad \text{on}\, y=0. \tag{7d}$$

We note that the system of equations and boundary conditions (7) involves only the single dimensionless parameter Pr and that the length of the plate l does not appear in the non-dimensionalization (6). l then enters the solution of our system in that range of interest is now

$$0\leqslant x\leqslant\frac{\nu^2 l}{g\beta\Delta T}\left(\frac{k_s}{bk_f}\right)^4. \tag{8}$$

However, the parabolic nature of the boundary-layer equations means that we require only one solution of equation (7) (for a given Pr), which can be obtained for all x, with the range of applicability for a given physical problem then given by equation (8).

3. SOLUTION

A solution of equation (7), valid for small x, can be obtained by writing

$$\psi=x^{4/5}F(x,\eta),\quad \theta=x^{1/5}H(x,\eta),\quad \eta=y/x^{1/5}. \tag{9a}$$

Equation (7d) then gives

$$\frac{\partial H}{\partial\eta}=1+x^{1/5}H\quad \text{on}\ \eta=0 \tag{9b}$$

and a solution of the transformed equations can be obtained by expanding in powers of $x^{1/5}$. This is not pursued further here as it was the method used by Pozzi and Lupo [3]. They obtained a solution to a large number of terms in this expansion and then extrapolated these results to large x. We need note only the form of transformation (9a), which will be required for the finite-difference solution described below.

A solution, valid for x large, can also be obtained, for which we require the transformation

$$\psi=x^{3/4}f(x,\zeta),\quad \theta=\theta(\zeta,x),\quad \zeta=y/x^{1/4}. \tag{10}$$

It is worth mentioning that equation (10) is the transformation appropriate for the problem of free convection from a vertical flat plate with uniform surface temperature [11, 12]. Using (10), equations and boundary conditions (7) become

$$\frac{\partial^3 f}{\partial\zeta^3}+\theta+\frac{3}{4}f\frac{\partial^2 f}{\partial\zeta^2}-\frac{1}{2}\left(\frac{\partial f}{\partial\zeta}\right)^2=x\left(\frac{\partial f}{\partial\zeta}\frac{\partial^2 f}{\partial\zeta\,\partial x}-\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial\zeta^2}\right) \tag{11a}$$

$$\frac{1}{Pr}\frac{\partial^2\theta}{\partial\zeta^2}+\frac{3}{4}f\frac{\partial\theta}{\partial\zeta}=x\left(\frac{\partial f}{\partial\zeta}\frac{\partial\theta}{\partial x}-\frac{\partial f}{\partial x}\frac{\partial\theta}{\partial\zeta}\right) \tag{11b}$$

$$f=0,\quad \frac{\partial f}{\partial\zeta}=0,\quad \theta-1=x^{-1/4}\frac{\partial\theta}{\partial\zeta}\quad \text{on}\ \zeta=0 \tag{11c}$$

$$\frac{\partial f}{\partial\zeta}\to 0,\quad \theta\to 0\quad \text{as}\ \zeta\to\infty. \tag{11d}$$

Equation (11c) suggests looking for a solution by expanding

$$\begin{aligned} f(x,\zeta) &= f_0(\zeta)+x^{-1/4}f_1(\zeta)\\ &\quad +x^{-1/2}f_2(\zeta)+x^{-3/4}f_3(\zeta)+\dots\\ \theta(x,\zeta) &= \theta_0(\zeta)+x^{-1/4}\theta_1(\zeta)\\ &\quad +x^{-1/2}\theta_2(\zeta)+x^{-3/4}\theta_3(\zeta)+\dots \end{aligned} \tag{12}$$

At leading order, the equations satisfied by (f_0,θ_0) are those for a uniform plate temperature, the solution of which is well documented [1, 11, 12] (and depends on Pr). At $0(x^{-1/4})$ we obtain the linear system

$$f_1'''+\theta_1+\tfrac{3}{4}f_0f_1''-\tfrac{3}{4}f_0'f_1'+\tfrac{1}{2}f_0''f_1=0 \tag{13a}$$

$$\frac{1}{Pr}\theta_1''+\tfrac{3}{4}f_0\theta_1'+\tfrac{1}{2}\theta_0'f_1+\tfrac{1}{4}f_0'\theta_1=0 \tag{13b}$$

subject to boundary conditions

$$\begin{aligned} &f_1(0)=0,\quad f_1'(0)=0,\quad \theta_1(0)=\theta_0'(0)\\ &f_1'\to 0,\quad \theta_1\to 0\quad \text{as}\ \zeta\to\infty, \end{aligned} \tag{13c}$$

where primes denote differentiation with respect of ζ. Note that the leading order solution appears both in equation (13a,b) and in boundary condition (13c), and if we just require the wall temperature, i.e. $\theta_1(0)$, this can be obtained directly from equation (13c) without solving equation (12).

The solution for the higher order terms can be continued in this way. However, we find that at $0(x^{-1})$, the first eigensolution

$$f_e=\zeta f_0'-3f_0\quad \theta_e=\zeta\theta_0' \tag{14}$$

arises due to the leading edge shift effect [13]. This requires the inclusion of term of $0(x^{-1}\log x)$ and an indeterminacy in the expansion arises (arbitrary multiples of (f_e,θ_e) can be added to the solution at $0(x^{-1})$). Hence the usefulness of asymptotic expansion (12) is confined to terms up to $0(x^{-3/4})$. The equations have to be solved numerically, and we find that the dimensionless wall temperature $\theta_w(x)\equiv\theta(x,0)$ (related to T_w by $T_w=\Delta T\theta_w+T_\infty$) is given, for x large, by

$$\theta_w(x) \sim 1-0.35683x^{-1/4}+0.12975x^{-1/2} -0.03521x^{-3/4}+\ldots \quad (15a)$$

for $Pr = 0.72$, and

$$\theta_w(x) \sim 1-0.74551x^{-1/4}+0.57699x^{-1/2} -0.34319x^{-3/4}+\ldots \quad (15b)$$

for $Pr = 7.0$.

To obtain a solution valid for all x, equations (7) were solved numerically. To do this we used the method of continuous transformations proposed by Hunt and Wilks [14]. This method incorporates an initial transformation of variables which reflects the form of the solution for both x small and x large. Following ref. [14] and using equations (9a) and (10), the transformation

$$\psi = \xi^4(1+\xi^5)^{-1/20}\phi(\xi, Y)$$
$$\theta = \xi(1+\xi^5)^{-1}G(\xi, Y)$$
$$Y = y\xi^{-1}(1+\xi^5)^{-1/20}, \quad \xi = x^{1/5} \quad (16)$$

is suggested. Note that equation (16) reduces to equation (9a) for small x and reduces to equation (10) when x is large. Using equation (16) equations (7) become

$$\frac{\partial^3\phi}{\partial Y^3}+G+\frac{1}{20}\left(\frac{16+15\xi^5}{1+\xi^5}\right)\phi\frac{\partial^2\phi}{\partial Y^2} -\frac{1}{10}\left(\frac{6+5\xi^5}{1+\xi^5}\right)\left(\frac{\partial\phi}{\partial Y}\right)^2 = \frac{\xi}{5}\left(\frac{\partial\phi}{\partial Y}\frac{\partial^2\phi}{\partial Y\partial\xi}-\frac{\partial\phi}{\partial\xi}\frac{\partial^2\phi}{\partial Y^2}\right) \quad (17a)$$

$$\frac{1}{Pr}\frac{\partial^2 G}{\partial Y^2}+\frac{1}{20}\left(\frac{16+15\xi^5}{1+\xi^5}\right)\phi\frac{\partial G}{\partial Y}-\frac{1}{5}\frac{1}{(1+\xi^5)}G\frac{\partial\phi}{\partial Y} = \frac{\xi}{5}\left(\frac{\partial\phi}{\partial Y}\frac{\partial G}{\partial\xi}-\frac{\partial\phi}{\partial\xi}\frac{\partial G}{\partial Y}\right) \quad (17b)$$

subject to the boundary conditions that

$$\phi = 0, \quad \frac{\partial\phi}{\partial Y} = 0$$
$$\frac{\partial G}{\partial Y} = (1+\xi^5)^{1/20}\xi G-(1+\xi^5)^{1/4} \quad \text{on } Y = 0$$
$$\frac{\partial\phi}{\partial Y}\to 0, \quad G\to 0 \quad \text{as } Y\to\infty. \quad (17c)$$

The dimensionless wall temperature, θ_w, and the dimensionless skin friction at the wall, $\tau_w = (\partial u/\partial y)_{y=0}$, are given by

$$\theta_w = \xi(1+\xi^5)^{-1}G(\xi, 0) \quad (18a)$$

$$\tau_w = \xi^2(1+\xi^5)^{-3/20}\frac{\partial^2\phi}{\partial Y^2}(\xi, 0). \quad (18b)$$

A finite-difference scheme was derived from equations (17) along the lines described in refs. [5–7]. Derivatives in the ξ-direction were differenced and all other terms averaged over the step from $\xi = \xi_1$ to $\xi = \xi_1+\Delta\xi$. This resulted in a pair of coupled nonlinear ordinary differential equations, which were then written in difference form.

This followed closely the method used previously [5–7], except that more care had to be taken to build the boundary condition on G on $Y = 0$ into the finite-difference scheme. To do this we applied equation (17b) on $Y = 0$, which gave (in difference form)

$$U_1-2U_0+U_{-1} = 0, \quad (19a)$$

where $U_0 = G(\xi_1, 0)+G(\xi_1+\Delta\xi_1, 0)$ and $U_{\pm 1} = G(\xi_1, \pm\Delta Y)+G(\xi_1+\Delta\xi, \pm\Delta Y)$. Boundary condition (17c) was then differenced to give

$$U_{-1} = U_1-(2\xi_1+\Delta\xi)\left(1+\frac{(2\xi_1+\Delta\xi)^5}{32}\right)^{1/20}U_0 +2h\left(1+\frac{(2\xi_1+\Delta\xi)^5}{32}\right)^{1/4}. \quad (19b)$$

This expression for U_{-1} was then substituted into equation (19a) which formed the first equation in the difference form of equation (17b).

All of this results in two sets of nonlinear algebraic equations which were solved iteratively using the Newton–Raphson method, taking as an initial estimate the profile at the previous step in ξ. At each iteration, the algebraic equations were solved using Choleski decomposition [15]. The iterations were found to converge quickly, taking typically no more iterations than in the previous cases treated, where either the plate temperature or heat flux were prescribed. Hence the modification to the boundary conditions given in equation (17c) appears to present no further computational problems. In this way an efficient numerical algorithm was devised which advanced the solution from $\xi = 0$ up to large values of ξ, using relatively small amounts of computer time and storage.

Finally, a check was made on the error from differentiating in the ξ-direction by covering the step ξ_1 to $\xi_1+\Delta\xi$ in first one, and then two steps, and insisting that the difference between the two solutions at $\xi_1+\Delta\xi$ was less than 5.10^{-5}. A step $\Delta Y = 0.05$ was used for the calculations with one outer boundary condition applied on $Y = Y_\infty = 15$ for $Pr = 0.72$ and $Y_\infty = 20$ for $Pr = 7.0$.

4. RESULTS AND DISCUSSION

We start by comparing the results for θ_w and τ_w obtained from our finite-difference solution, with those obtained from the series expansion given by Pozzi and Lupo [3]. The results for $Pr = 0.733$ are shown in Table 1; the coefficients in the series quoted in ref. [3] are given to four figures and consequently

Table 1. Comparison of values for the dimensionless wall temperature θ_w and skin friction τ_w obtained from the numerical solution and the series solution of Pozzi and Lupo [3] for $Pr = 0.733$

	θ_w		τ_w	
ξ	Present, equation (18a)	Series [3]	Present, equation (18b)	Series [3]
0.1	0.177	0.177	0.014	0.014
0.2	0.310	0.310	0.051	0.051
0.3	0.413	0.413	0.105	0.105
0.4	0.493	0.493	0.172	0.172
0.5	0.557	0.557	0.250	0.250
0.6	0.608	0.608	0.337	0.337
0.7	0.651	0.651	0.430	0.430
0.8	0.686	0.684	0.530	0.530
0.9	0.715	0.708	0.635	0.635
1.0	0.741	0.717	0.745	0.741
1.1	0.762	0.699	0.859	0.829
1.2	0.781	0.640	0.972	0.817

the values obtained from summing the series will be reliable only to three decimal places. From this table we can see that values for both the wall temperature θ_w and skin friction τ_w obtained by the two methods agree (to three decimal places) up to $\xi = 0.7$ (for θ_w) and $\xi = 0.9$ (for τ_w). Thereafter the two sets of results increasingly diverge with values for both θ_w and τ_w starting to decrease. This is contrary to both the finite-difference solution, which shows a monotone increase in both θ_w and τ_w, and to the asymptotic series equation (15). Pozzi and Lupo [3] estimated the radius of convergence x_0 of their series expansion as $x_0 \simeq 1.16$ ($\xi \simeq 1.03$) and the results shown in Table 1 appear to confirm this prediction.

Graphs of the non-dimensional wall temperature $\theta_w(x)$ against x are shown in Fig. 2 (for $Pr = 0.72$) and in Fig. 3 (for $Pr = 7.0$). Also shown in these figures (by the broken lines) are the values of θ_w as obtained from the asymptotic expansions (15a) and (15b), respectively. The asymptotic limit, $\theta_w \to 1$ as $x \to \infty$, is also shown (by the dot lines). From these figures it appears that the asymptotic expansion for x large, gives a better representation for θ_w for the smaller value of Pr, with the difference between the asymptotic series and the numerically determined values even at $x = 1$, being only 0.5% for $Pr = 0.72$, whereas, this difference is just over 16% for $Pr = 7.0$ at $x = 1$. Also, the asymptotic limit is approached more quickly for the smaller value of Pr.

In Fig. 4 we give plots of $\tau_w(x)$ against x for $Pr = 0.72$ and $Pr = 7.0$, respectively. Here we can see the $x^{2/5}$ singularity near $x = 0$, with the values for

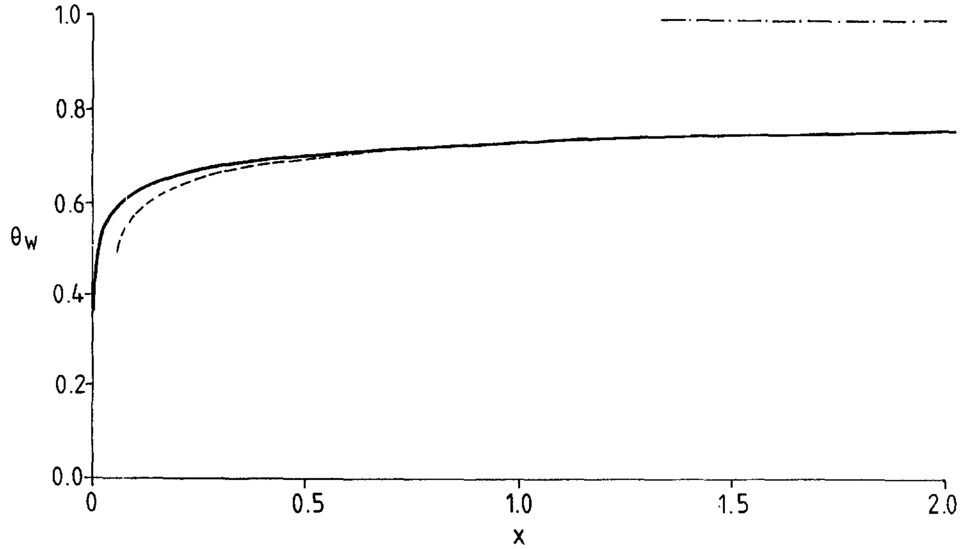

Fig. 2. The non-dimensional wall temperature θ_w plotted against x for $Pr = 0.72$. —— numerical; --- equation (15a); —·— $\theta_w \to 1$.

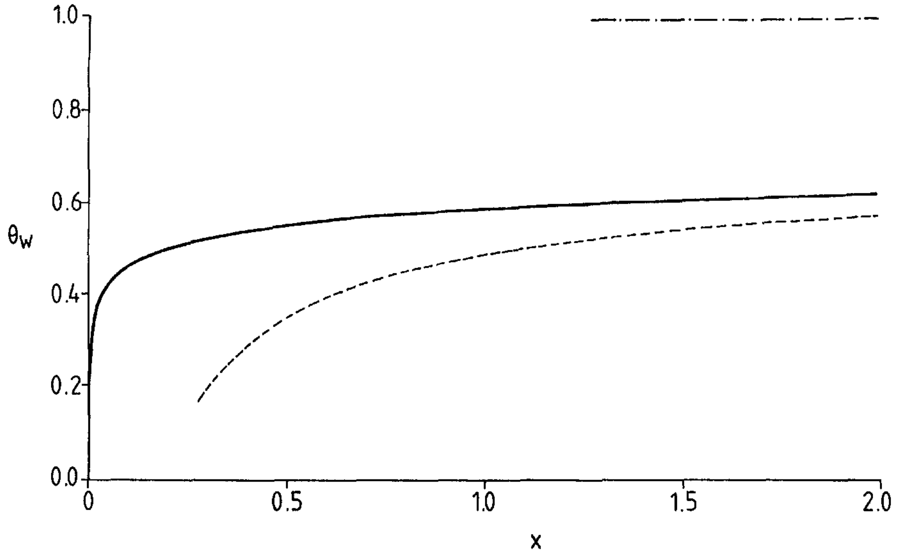

Fig. 3. The non-dimensional wall temperature θ_w plotted against x for $Pr = 7.0$. —— numerical; --- equation (15b); —·— $\theta_w \to 1$.

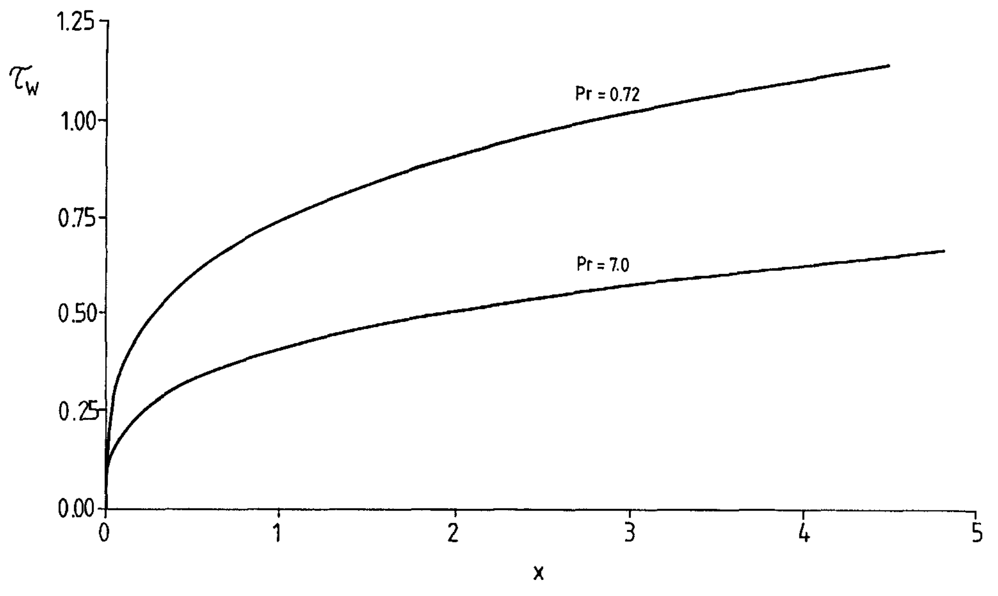

Fig. 4. Dimensionless skin friction τ_w plotted against x for $Pr = 0.72$ and $Pr = 7.0$.

$Pr = 0.72$ increasing more rapidly with x than for $Pr = 7.0$. A more detailed examination of the numerical results shows that the maximum value of u attained at a given value of x is greater for $Pr = 0.72$ than for $Pr = 7.0$.

Finally, in Figs. 5 and 6 we give temperature profiles θ at various values of x plotted against y for $Pr = 0.72$ and $Pr = 7.0$. These show the rise in the wall temperature as x is increased (in line with Figs. 2 and 3) and that the temperature profiles become more spread out as x is increased, with this spreading in y being greater for $Pr = 0.72$ than for $Pr = 7.0$.

5. CONCLUSION

We have been able to exploit the parabolic nature of the boundary-layer equations so as to reduce the problem of calculating the heat transfer and flow characteristics of the conjugate heat transfer on a vertical surface, to the solution of a set of equations

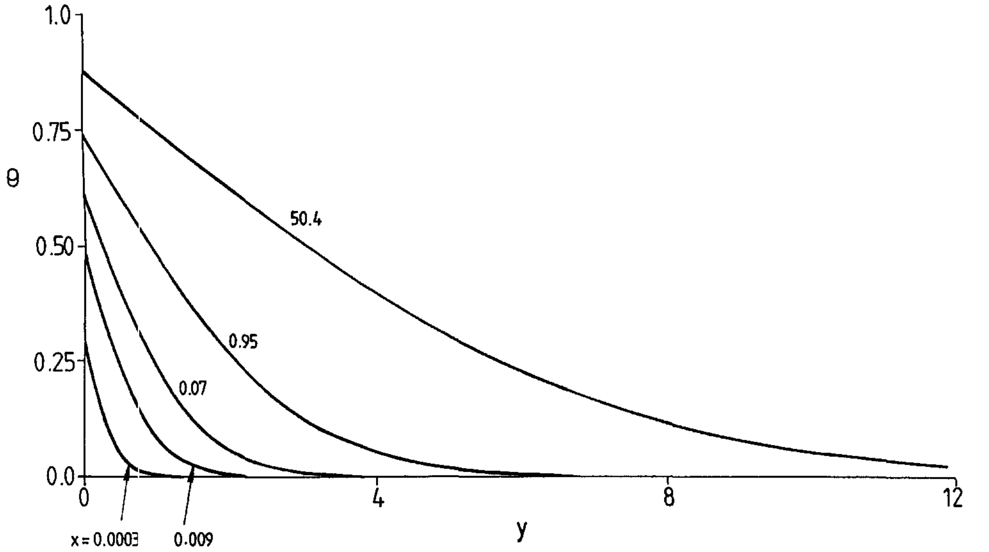

Fig. 5. Temperature profiles θ plotted against y for $Pr = 0.72$, at $x = 0.0003, 0.009, 0.07, 0.95, 50.4$.

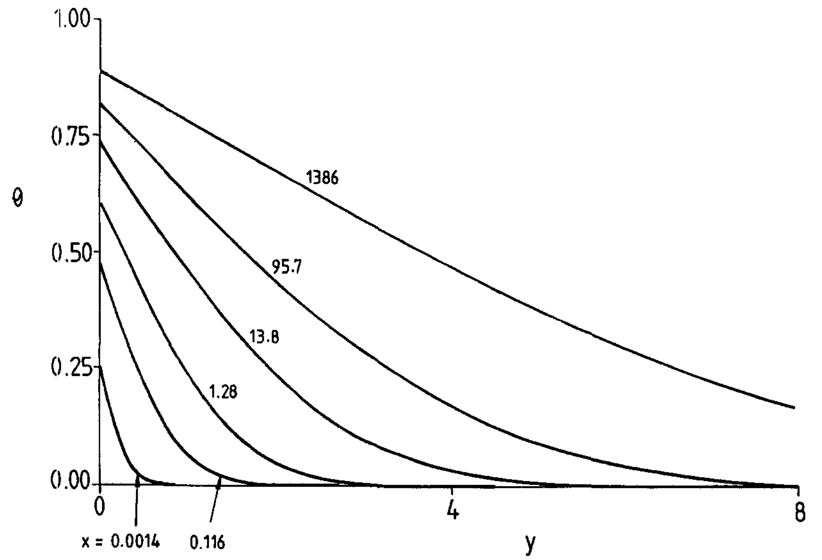

Fig. 6. Temperature profiles θ plotted against y for $Pr = 7.0$, at $x = 0.116, 1.28, 13.8, 95.7, 1386$.

dependent only on the Prandtl number. The range of applicability of the results for a given physical situation is then determined by equation (8). Also, we have presented an efficient finite-difference scheme for handling the radiation (or Robin) boundary condition (7d), which gives accurate results for all values of x without recourse to previously presented series extension methods. Moreover, we have noted that asymptotic expansions can be used to give a good representation of the heat transfer characteristics, even at relatively low values of x.

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